

# Singular Deformation Theory and the Invariance of Gerstenhaber Algebra Structure on the Singular Hochschild Cohomology

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## Abstract

Keller proved in 1999 that the Gerstenhaber algebra structure of the Hochschild cohomology of an algebra is an invariant of the derived category. In this paper, we adapt his approach and develop the singular infinitesimal deformation theory. As a consequence, we show that the Gerstenhaber algebra structure on the singular Hochschild cohomology of an algebra is preserved under singular equivalences of Morita type with level.

## 1 Introduction

In the recent paper [Wang2], we defined the singular Hochschild cohomology group  $\mathrm{HH}_{\mathrm{sg}}^i(A, A)$  of an associative algebra  $A$  over a field  $k$  as morphisms from  $A$  to  $A[i]$  in  $\mathcal{D}_{\mathrm{sg}}(A^{\mathrm{op}} \otimes_k A)$  for any  $i \in \mathbb{Z}$ . Similar to the case of the Hochschild cohomology ring  $\mathrm{HH}^*(A, A)$ , we proved that the singular Hochschild cohomology ring  $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$  has a Gerstenhaber algebra structure (cf. [Ger]). Moreover, the natural morphism

$$\mathrm{HH}^*(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A)$$

is a homomorphism of Gerstenhaber algebras.

Keller proved in [Kel1] that the Gerstenhaber algebra structures on Hochschild cohomology rings are preserved under derived equivalences of standard type. That is, let  $A$  and  $B$  be two  $k$ -algebras and if  $X$  is a complex of  $A$ - $B$ -bimodules such that the total derived tensor product by  $X$  is an equivalence  $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$  between the derived categories of  $A$  and of  $B$ , then  $X$  yields a natural isomorphism of Gerstenhaber algebras from  $\mathrm{HH}^*(A, A)$  to  $\mathrm{HH}^*(B, B)$ . In this paper, we will show that the Gerstenhaber algebra structure on the singular Hochschild cohomology ring is also preserved under derived equivalences of standard type. In fact, we will prove a stronger result, that is, the Gerstenhaber algebra structure on the singular Hochschild cohomology ring is preserved under singular equivalences of Morita type with level (cf. [Wang1] and Section 6 below). Recall that a derived equivalence of standard type induces a singular equivalence of Morita type with level (cf. [Wang1]).

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The reminder of this paper is organized as follows. In Section 2, we recall the bar resolution of an associative algebra  $A$  and give some natural lifts of elements in  $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$  along the bar resolution. The bullet product  $\bullet$  is the key ingredient of the Gerstenhaber algebra structure on the singular Hochschild cohomology. We will give a derived interpretation of this bullet product  $\bullet$  in Section 3. Section 4 is devoted to recall some notions on  $R$ -relative derived categories and  $R$ -relative derived tensor products. In Section 5, we will generalize the infinitesimal deformation theory in [Kel1] to what we call the singular infinitesimal deformation theory. As a result, we give an interpretation of the Gerstenhaber bracket on the singular Hochschild cohomology ring from the point of view of singular infinitesimal deformation theory. This interpretation relies on the derived interpretation of the bullet product  $\bullet$  in Section 3. In the last section, we will show our main theorem. Namely,

**Theorem 1.1** (=Theorem 6.3). *Let  $k$  be a field and let  $A$  and  $B$  be two finite dimensional  $k$ -algebras. Suppose that  $({}_A M_{B,B} N_A)$  defines a singular equivalence of Morita type with level  $l \in \mathbb{Z}_{\geq 0}$ . Then the functor*

$$M \otimes_B - \otimes_B N : \mathcal{D}_{\mathrm{sg}}(B \otimes B^{\mathrm{op}}) \rightarrow \mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$$

*induces an isomorphism of Gerstenhaber algebras between the singular Hochschild cohomology rings  $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$  and  $\mathrm{HH}_{\mathrm{sg}}^*(B, B)$ .*

Throughout this paper, we fix by  $k$  as the commutative base ring with a unit. For simplicity, the symbol  $\otimes$  always represents the tensor product  $\otimes_k$  over the base ring  $k$ . For a  $k$ -algebra  $A$ , we denote the element  $(a_i \otimes a_{i+1} \otimes \cdots \otimes a_j) \in A^{\otimes j-i+1} (i \leq j)$  sometimes by  $a_{i,j}$  for short. We frequently use some notions on differential graded algebras and relative derived categories in this paper, for more details, we refer to [Kel1, Kel2, BeLu]. We also refer to [KeVo, Ric, Wei, Zim] for some notions on triangulated categories and derived categories.

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## 2 Bar resolution of an associative $k$ -algebra

### 2.1 Definition

Let  $A$  be an associative algebra over a commutative ring  $k$  such that  $A$  is projective as a  $k$ -module. Then there is a projective resolution of  $A$  as an  $A$ - $A$ -bimodule,

$$\mathrm{Bar}_*(A) : \cdots \longrightarrow A^{\otimes r+2} \xrightarrow{d_r} A^{\otimes r+1} \xrightarrow{d_{r-1}} \cdots \longrightarrow A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \longrightarrow 0, \quad (1)$$

where the differential  $d_r : A^{\otimes r+2} \rightarrow A^{\otimes r+1}$  is defined by

$$d_r(a_1 \otimes \cdots \otimes a_{r+2}) := \sum_{i=1}^{r+1} (-1)^{i-1} a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,r+2},$$

for any  $a_1 \otimes \cdots \otimes a_{r+2} \in A^{\otimes r+2}$ . Here we denote, for  $i < j$ ,

$$a_{i,j} := a_i \otimes a_{i+1} \otimes \cdots \otimes a_j.$$

This is the so-called (un-normalized) bar resolution of  $A$ . We also have the normalized bar resolution  $\overline{\text{Bar}}_*$  of  $A$ . It is defined by

$$\overline{\text{Bar}}_r(A) := A \otimes \overline{A}^{\otimes r} \otimes A,$$

where  $\overline{A} = A/(k \cdot 1)$ , and with the induced differential from that of  $\text{Bar}_*(A)$ . Note that  $\overline{\text{Bar}}_*(A)$  is also a projective resolution of  $A$  as an  $A$ - $A$ -bimodule.

For any  $r \in \mathbb{Z}_{\geq 1}$ , let us denote by  $\Omega^r(A)$ , the image of the differential

$$d_r : A^{\otimes r+2} \rightarrow A^{\otimes r+1}$$

in the un-normalized bar resolution  $\text{Bar}_*(A)$ . Similarly, we denote by  $\overline{\Omega}^r(A)$ , the image of the differential

$$\overline{d}_r : A \otimes \overline{A}^{\otimes r} \otimes A \rightarrow A \otimes \overline{A}^{\otimes r-1} \otimes A$$

in the normalized bar resolution  $\overline{\text{Bar}}_*(A)$ . In particular, we will use the notation

$$\Omega^0(A) = \overline{\Omega}^0(A) := A.$$

Observe that we have the un-normalized bar resolution  $\text{Bar}_*(\Omega^r(A))$  of the  $A$ - $A$ -bimodule  $\Omega^r(A)$ :

$$\text{Bar}_*(\Omega^r(A)) : \cdots \longrightarrow A^{\otimes r+s+2} \xrightarrow{d_{r+s}} A^{\otimes r+s+1} \xrightarrow{d_{r+s-1}} \cdots \longrightarrow A^{\otimes r+2} \longrightarrow 0$$

and the normalized bar resolution  $\overline{\text{Bar}}_*(\overline{\Omega}^r(A))$  of the  $A$ - $A$ -bimodule  $\overline{\Omega}^r(A)$ :

$$\overline{\text{Bar}}_*(\overline{\Omega}^r(A)) : \cdots \longrightarrow A \otimes \overline{A}^{\otimes r+s} \otimes A \xrightarrow{\overline{d}_{r+s}} A \otimes \overline{A}^{\otimes r+s-1} \otimes A \xrightarrow{\overline{d}_{r+s-1}} \cdots \longrightarrow A \otimes \overline{A}^{\otimes r} \otimes A \longrightarrow 0$$

In the rest of this paper, let us just consider the un-normalized bar resolution. For the case of normalized bar resolution, we always have the analogous results.

**Remark 2.1.** Note that the following chain complex

$$(\text{Bar}_*(\Omega^p(A)), d_p) := \text{Bar}_*(\Omega^p(A)) \xrightarrow{d_p} \Omega^p(A)$$

is exact for any fixed  $p \in \mathbb{Z}_{\geq 0}$ . In this remark, we will show that the identity morphism on  $(\text{Bar}_*(\Omega^p(A)), d_p)$  is homotopy equivalent to zero, where we use the homotopy

$$s_p : \text{Bar}_r(\Omega^p(A)) \rightarrow \text{Bar}_{r+1}(\Omega^p(A))$$

$$s_p(x) := (-1)^{rp} x \otimes 1.$$

Namely, we have the following diagram.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{\otimes p+3} & \xrightarrow{d_{p+1}} & A^{\otimes p+2} & \xrightarrow{d_p} & \Omega^p(A) \longrightarrow 0 \\ & & \downarrow \text{id} & \swarrow s_p & \downarrow \text{id} & \swarrow s_p & \downarrow \text{id} \\ \cdots & \longrightarrow & A^{\otimes p+3} & \xrightarrow{d_{p+1}} & A^{\otimes p+2} & \xrightarrow{d_p} & \Omega^p(A) \longrightarrow 0 \end{array}$$

It is straightforward to verify that

$$s_p \circ d + d \circ s_p = \text{id}.$$

So it follows that the identity morphism is homotopy equivalent to zero on  $(\text{Bar}_*(\Omega^p(A)), d_p)$ . Note that  $s_p$  is a morphism of graded left  $A$ -modules rather than a morphism of graded  $(A \otimes A^{\text{op}})$ -modules. Therefore the complex  $(\text{Bar}_*(\Omega^p(A)), d_p)$  vanishes in the homotopy category  $\mathcal{K}^-(A\text{-mod})$  and does not, in general in the homotopy category  $\mathcal{K}^-(A \otimes A^{\text{op}}\text{-mod})$ .

For any  $p, q \in \mathbb{Z}_{\geq 0}$ , we will construct a morphism of complexes of  $A$ - $A$ -bimodules between  $\text{Bar}_*(\Omega^{p+q}(A))$  and  $\text{Bar}_*(\Omega^p(A)) \otimes_A \text{Bar}_*(\Omega^q(A))$ . Recall that for any  $r \in \mathbb{Z}_{\geq 0}$ , we have

$$\begin{aligned} \text{Bar}_r(\Omega^p(A)) &= A^{\otimes p+r+2}, \\ (\text{Bar}_*(\Omega^p(A)) \otimes_A \text{Bar}_*(\Omega^q(A)))_r &= \bigoplus_{i=0}^r \text{Bar}_i(\Omega^p(A)) \otimes_A \text{Bar}_{r-i}(\Omega^q(A)). \end{aligned}$$

Let us define

$$\Delta_{p,q} : \text{Bar}_*(\Omega^{p+q}(A)) \rightarrow \text{Bar}_*(\Omega^p(A)) \otimes_A \text{Bar}_*(\Omega^q(A)) \quad (2)$$

as follows. For any  $r \in \mathbb{Z}_{\geq 0}$  and  $a_{1,p+q+r+2} \in \text{Bar}_r(\Omega^{p+q}(A))$ , define

$$\Delta_{p,q;r}(a_{1,p+q+r+2}) := \sum_{i=0}^r (a_{1,p+i+1} \otimes 1) \otimes_A (1 \otimes a_{p+i+2,p+q+2}).$$

Then it is straightforward to verify that  $\Delta_{p,q}$  is a well-defined homomorphism of chain complexes of  $A$ - $A$ -bimodules.

**Lemma 2.2.** *If  $pq=0$ , then  $\Delta_{p,q}$  is an isomorphism of  $\mathcal{K}(A \otimes A^{\text{op}})$ .*

*Proof.* Note that  $\Delta_{0,q}$  is a lift of the identity morphism  $\text{id}_{\Omega^q(A)} : \Omega^q(A) \rightarrow \Omega^q(A)$ . Hence  $\Delta_{0,q}$  is a quasi-isomorphism. Since  $\text{Bar}_*(\Omega^q(A))$  is a complex of projective  $A$ - $A$ -bimodules, it follows that  $\Delta_{0,q}$  is also an isomorphism in  $\mathcal{K}(A \otimes A^{\text{op}})$ . By the same argument as above, we have that  $\Delta_{p,0}$  is an isomorphism of  $\mathcal{K}(A \otimes A^{\text{op}})$ . ■

**Remark 2.3.**  $\Delta_{p,q}$  plays a quite important rôle in our discussion below. Recall that  $\Delta_{0,0}$  is induced from the coproduct  $\Delta$  in the tensor coalgebra

$$T^c A := \bigoplus_{n \geq 0} A^{\otimes n}$$

defined in [Sta] and [Kel1].

## 2.2 Two types of lift

Let  $A$  be an associative  $k$ -algebra such that  $A$  is projective as a  $k$ -module. Let  $m, p \in \mathbb{Z}_{\geq 0}$  and  $f \in \text{HH}^m(A, \Omega^p(A))$ . By definition,  $f$  can be represented by an element

$$f \in \text{Hom}_k(A^{\otimes m}, \Omega^p(A))$$

such that

$$\delta(f) = 0,$$

where  $\delta$  is the differential of the Hochschild cochain complex  $\text{Hom}_k(A^{\otimes*}, \Omega^p(A))$ . Denote by  $\epsilon_m$  the graded vector space defined as follows,

$$(\epsilon_m)_i := \begin{cases} k & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following two types of lift in the category  $C(A \otimes_k A^{\text{op}})$  of chain complexes of  $A$ - $A$ -bimodules.

$$\begin{aligned} \theta_1(f) : \text{Bar}_*(A) &\rightarrow \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1], \\ \theta_2(f) : \text{Bar}_*(A) &\rightarrow \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1], \end{aligned}$$

where  $\theta_1(f)$  and  $\theta_2(f)$  are defined, respectively, as follows,

$$\begin{aligned} \theta_1(f)(a_{1,m+r+2}) &:= (-1)^{p+1+(m-p-1)r} a_1 f(a_{2,m+1}) \otimes a_{m+2,m+r+2}, \\ \theta_2(f)(a_{1,m+r+2}) &:= (-1)^{r+1} a_{1,r+1} \otimes f(a_{r+2,r+m+1}) a_{r+m+2}. \end{aligned}$$

Note that  $\theta_1(f)$  and  $\theta_2(f)$  are indeed morphisms of chain complexes of  $A$ - $A$ -bimodules. It is well-known in homological algebra (cf. e.g. [Wei, Comparison Theorem 2.2.6]) that  $\theta_1(f)$  and  $\theta_2(f)$  are homotopy equivalent to each other. Moreover, there exists a specific chain homotopy

$$s(f) : \text{Bar}_*(A) \rightarrow \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}$$

from  $\theta_1(f)$  to  $\theta_2(f)$ , defined as follows: for any  $r \in \mathbb{Z}_{\geq 0}$ ,

$$s^r(f) : A^{\otimes m+r+2} \rightarrow A^{\otimes p+r+3} \quad (3)$$

sends  $a_{1,m+r+2} \in A^{\otimes m+r+2}$  to

$$\sum_{i=1}^r (-1)^{i-1} a_{1,i} \otimes f(a_{i+1,i+m+1}) \otimes a_{i+m+2,m+r+2}.$$

Namely, we have the following diagram,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{\otimes m+3} & \xrightarrow{d_{m+1}} & A^{\otimes m+2} & \xrightarrow{d_m} & A^{\otimes m+1} \xrightarrow{d_{m-1}} \cdots \\ & & \downarrow \theta_1(f) & \nearrow s^0(f) & \downarrow \theta_1(f) & \nearrow 0 & \downarrow \\ \cdots & \longrightarrow & A^{\otimes p+3} & \xrightarrow{-d_{p+1}} & A^{\otimes p+2} & \longrightarrow & 0 \end{array}$$

which means that

$$\theta_1(f) - \theta_2(f) = s(f) \circ d + d \circ s(f). \quad (4)$$

We will denote the  $r$ -th lift of  $\theta_1(f)$  by  $\Omega^r(\theta_1(f))$ , namely,

$$\begin{aligned} \Omega^r(\theta_1(f)) : A^{\otimes m+r+2} &\rightarrow \Omega^{p+r}(A) \\ a_{1,m+r+2} &\mapsto (-1)^{p+(m-p)r} d(a_1 f(a_{2,m+1}) \otimes a_{m+2,m+r+2}). \end{aligned}$$

Similarly, we denote the  $r$ -th lift of  $\theta_2(f)$  by  $\Omega^r(\theta_2(f))$ ,

$$\begin{aligned} \Omega^r(\theta_2(f)) : A^{\otimes m+r+2} &\rightarrow \Omega^{p+r}(A) \\ a_{1,m+r+2} &\mapsto d(a_{1,r+1} \otimes f(a_{r+2,r+m+1}) a_{r+m+2}). \end{aligned}$$

**Remark 2.4.** Since  $s(f)$  is a morphism of graded  $(A^{\text{op}} \otimes A)$ -modules, it follows that

$$\theta_1(f) = \theta_2(f)$$

in the homotopy category  $\mathcal{K}^-(A^{\text{op}} \otimes A\text{-mod})$  and both of them represent the same element  $f \in \text{Hom}_{\mathcal{D}^b(A^{\text{op}} \otimes A)}(A, \Omega^p(A)[m])$ .

### 3 A derived interpretation of the bullet product •

#### 3.1 Definition of $R$ -relative derived category

Let us start with the general setting. Let  $R$  be a commutative differential graded  $k$ -algebra and  $E$  be a differential graded  $R$ -algebra. The  $R$ -relative (unbounded) derived category  $\mathcal{D}_R(E)$  is a  $k$ -linear category with objects differential graded  $E$ -modules and morphisms obtained from morphisms of differential graded  $E$ -modules by formally inverting all  $R$ -relative quasi-isomorphisms, i.e. all morphisms  $s : L \rightarrow M$  of differential graded  $E$ -modules whose restriction to  $R$  is a homotopy equivalence. For example, the  $k$ -relative derived category  $\mathcal{D}_k(E)$  is exactly the usual derived category  $\mathcal{D}(E)$  of the differential graded algebra  $E$ . The  $R$ -relative derived category  $\mathcal{D}_R(R)$  is just the homotopy category  $K(R)$  of  $R$ . For more details on  $R$ -relative derived categories, we refer to [Kel1, Kel2]. For convenience, from now on, the term modules refers to right modules.

**Remark 3.1.** We also consider the  $R$ -relative bounded derived category  $\mathcal{D}_R^b(E)$ , which is by definition, the full subcategory of  $\mathcal{D}_R(E)$  consisting of those objects  $X$  such that there are only finitely many integers  $i$  such that  $H_i(X) \neq 0$ .

Let  $A$  be an associative algebra over a commutative ring  $k$  such that  $A$  is projective as a  $k$ -module and  $R$  be any commutative differential graded  $k$ -algebra. Then  $A^{\text{op}} \otimes_k A \otimes_k R$  is a differential graded  $R$ -algebra.

We denote by  $R_i$  the commutative differential graded algebra  $k[\epsilon_i]/(\epsilon_i^2)$ , where  $\epsilon_i$  is of degree  $i$  and the differential  $d = 0$ . We also denote by  $\epsilon_i$  the kernel of the augmentation  $R_i \rightarrow k$ . Let us consider the  $R_i$ -relative derived category  $\mathcal{D}_{R_i}(A^{\text{op}} \otimes_k A \otimes_k R_i)$ .

Let  $\alpha : X \rightarrow Y$  be a morphism of complexes. Let us construct a complex  $C(\alpha)$  associated to  $\alpha$ . As graded spaces,

$$C(\alpha) := X \oplus Y[-1]$$

and the differential is  $\begin{pmatrix} d_X & \alpha \\ 0 & d_Y[-1] \end{pmatrix}$ . For simplicity, we use the following diagram to represent the complex  $C(\alpha)$ :

$$\begin{array}{ccc} & \alpha & \\ X & \xrightarrow{\quad} & Y[-1], \\ & \oplus & \end{array}$$

Let  $m, p \in \mathbb{Z}_{\geq 0}$  and  $f \in \text{HH}^m(A, \Omega^p(A))$ . Recall that in Section 2.2 we defined two lifts  $\theta_1(f)$  and  $\theta_2(f)$  associated to  $f$ . Now let us construct two differential graded right  $(A^{\text{op}} \otimes_k A \otimes_k R_{m-p-1})$ -modules  $C(\theta_1(f))$  and  $C(\theta_2(f))$  associated to  $\theta_1(f)$  and  $\theta_2(f)$ , respectively. As a graded right  $(A^{\text{op}} \otimes_k A \otimes_k R_{m-p-1})$ -module,

$$C(\theta_1(f)) := \bigoplus_{i=0}^{p-1} (A^{\otimes i+2} \otimes k[i]) \bigoplus \text{Bar}_*(\Omega^p(A)) \otimes R_{m-p-1}[p] \quad (5)$$

where  $k$  is considered as a right  $R_{m-p-1}$ -module, and the differential is illustrated as follows,

$$\begin{array}{ccc} & \theta_1(f) & \\ \text{Bar}_*(A) & \xrightarrow{\quad} & \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}, \\ & \oplus & \end{array}$$

where we used the fact that

$$C(\theta_1(f)) = \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}$$

as graded  $k$ -modules. Similarly, as a graded right  $(A^{\text{op}} \otimes_k A \otimes_k R_{m-p-1})$ -module,

$$C(\theta_2(f)) := \bigoplus_{i=0}^{p-1} (A^{\otimes i+2} \otimes k[i]) \bigoplus \text{Bar}_*(\Omega^p(A)) \otimes R_{m-p-1}[p] \quad (6)$$

and the differential is illustrated as follows,

$$\begin{array}{ccc} & \theta_2(f) & \\ & \curvearrowright & \\ \text{Bar}_*(A) & \oplus & \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}. \end{array}$$

**Remark 3.2.** Denote

$$C(f) := \begin{array}{ccc} & f & \\ & \curvearrowright & \\ \text{Bar}_*(A) & \oplus & \Omega^p(A) \otimes \epsilon_{m-1}. \end{array}$$

Then we claim that  $C(f)$  is also a differential graded right  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module. Indeed, as a graded right  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module,

$$C(f) \cong \bigoplus_{i \neq p} (A^{\otimes i+2}[i] \otimes k) \bigoplus (A^{\otimes p+2}[p] \oplus \Omega^p(A)[m-1])$$

where  $(A^{\otimes p+2}[p] \oplus \Omega^p(A)[m-1])$  is a graded right  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module defined as follows,

$$x \cdot \epsilon_{m-p-1} := dx \in \Omega^p(A)[m-1]$$

for any  $x \in A^{\otimes p+2}[p]$ .

**Lemma 3.3.** *Let  $m, p \in \mathbb{Z}_{\geq 0}$  and  $f \in \text{HH}^m(A, \Omega^p(A))$ , then we have the following isomorphism in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ ,*

$$C(\theta_1(f)) \cong C(\theta_2(f)) \cong C(f).$$

*Proof.* Let us prove the first isomorphism. Consider the following morphism of chain complexes,

$$\begin{pmatrix} \text{id} & s(f) \\ 0 & \text{id} \end{pmatrix} : C(\theta_1(f)) \rightarrow C(\theta_2(f))$$

where  $s(f)$  is the chain homotopy defined in (3). Clearly, it is also a morphism of differential graded right  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -modules. On the other hand, we note that the morphism  $\begin{pmatrix} \text{id} & s(f) \\ 0 & \text{id} \end{pmatrix}$  is an isomorphism of dg right  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -modules with inverse

$$\begin{pmatrix} \text{id} & -s(f) \\ 0 & \text{id} \end{pmatrix} : C(\theta_2(f)) \rightarrow C(\theta_1(f)).$$

Hence we have an isomorphism in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ ,

$$C(\theta_1(f)) \cong C(\theta_2(f)).$$

It remains to verify the second isomorphism. Consider the following morphism of chain complexes of right  $A^{\text{op}} \otimes A$ -modules,

$$C(\theta_2(f)) \xrightarrow{\widehat{\sigma}_p := \begin{pmatrix} \text{id} & 0 \\ 0 & \sigma_p \end{pmatrix}} C(f)$$

where

$$\sigma_p : \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \rightarrow \Omega^p(A) \otimes \epsilon_{m-1}$$

is the canonical surjection induced by the surjection  $d_p : A^{p+2} \rightarrow \Omega^p(A)$ . Note that it is also a morphism of differential graded  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -modules. Let us consider the mapping cone  $\text{Cone}(\widehat{\sigma}_p)$ . We claim that

$$\text{Cone}(\widehat{\sigma}_p) = 0$$

in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ , equivalently,  $\text{Cone}(\widehat{\sigma}_p)$  is isomorphic to zero in the category  $\mathcal{K}(R_{m-p-1})$ . Indeed, let us write  $\text{Cone}(\widehat{\sigma}_p)$  as follows,

$$\text{Cone}(\widehat{\sigma}_p) \cong \text{Bar}_*(A)[1] \oplus \text{Bar}_*(A) \oplus (\text{Bar}_*(\Omega^p(A)), d_p) \otimes \epsilon_{m-p-1}[1]$$

$\xrightarrow{\theta_2(f)}$  (top arrow from  $\text{Bar}_*(A)[1]$  to  $(\text{Bar}_*(\Omega^p(A)), d_p) \otimes \epsilon_{m-p-1}[1]$ )  
 $\xrightarrow{f}$  (bottom arrow from  $\text{Bar}_*(A)$  to  $(\text{Bar}_*(\Omega^p(A)), d_p) \otimes \epsilon_{m-p-1}[1]$ )

Then we have the following homotopy,

$$\text{Cone}(\widehat{\sigma}_p) \xrightarrow{\widehat{s}_p} \text{Cone}(\widehat{\sigma}_p)[-1]$$

where  $\widehat{s}_p$  is defined as follows

$$\widehat{s}_p := \begin{pmatrix} s_0 & 0 & 0 \\ 0 & s_0 & f \otimes 1 \\ 0 & 0 & s_p \end{pmatrix}$$

where  $s_p$  is defined in Remark 2.1 and

$$(f \otimes 1)(a_{1,r}) = \begin{cases} a_1 f(a_{2,m+1}) \otimes 1 & \text{if } r = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $\widehat{s}_p$  is a well-defined homotopy of differential graded  $R_{m-p-1}$ -modules, so that the identity on  $\text{Cone}(\widehat{\sigma}_p)$  is homotopy equivalent to 0.

$$\widehat{s}_p \circ d_{\text{Cone}(\widehat{\sigma}_p)} + d_{\text{Cone}(\widehat{\sigma}_p)} \circ \widehat{s}_p = \text{id}_{\text{Cone}(\widehat{\sigma}_p)}.$$

So we have

$$\text{Cone}(\widehat{\sigma}_p) = 0$$

in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$  and thus  $\widehat{\sigma}_p$  is an  $R_{m-p-1}$ -relative quasi-isomorphism. Therefore, we have the following isomorphisms in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ ,

$$C(\theta_1(f)) \cong C(\theta_2(f)) \cong C(f).$$

■



### 3.2 A derived interpretation

First, let us recall the bullet product  $\bullet$  defined in [Wang2]. For any  $m \in \mathbb{Z}_{>0}$  and  $p \in \mathbb{Z}_{\geq 0}$ , denote

$$C^m(A, \Omega^p(A)) := \text{Hom}_k(A^{\otimes m}, \Omega^p(A)).$$

Then we have a double complex  $C^*(A, \Omega^*(A))$  with the horizontal map  $\delta$  induced from the differential of the bar resolution, and the vertical map zero.

Give  $f \in C^m(A, \Omega^p(A))$  and  $g \in C^n(A, \Omega^q(A))$ , denote

$$f \bullet_i g := \begin{cases} d((f \otimes \text{id}^{\otimes q})(\text{id}^{\otimes i-1} \otimes g \otimes \text{id}^{\otimes m-i}) \otimes 1) & \text{if } 1 \leq i \leq m, \\ d((\text{id}^{\otimes -i} \otimes f \otimes \text{id}^{\otimes q+i})(g \otimes \text{id}^{\otimes m-1}) \otimes 1) & \text{if } -q \leq i \leq -1, \end{cases}$$

where  $d$  is the differential of the bar resolution  $\text{Bar}_*(A)$ , and

$$f \bullet g := \sum_{i=1}^m (-1)^{p+q+(i-1)(q-n-1)} f \bullet_i g + \sum_{i=1}^q (-1)^{p+q+i(p-m-1)} f \bullet_{-i} g$$

Then we define

$$[f, g] := f \bullet g - (-1)^{(m-p-1)(n-q-1)} g \bullet f. \quad (7)$$

Note that

$$[f, g] \in C^{m+n-1}(A, \Omega^{p+q}(A)).$$

Then from Proposition 4.6 in [Wang2], it follows that  $[\cdot, \cdot]$  defines a differential graded Lie algebra structure on the total complex

$$\bigoplus_{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0}} C^m(A, \Omega^p(A))$$

and thus  $[\cdot, \cdot]$  defines a graded Lie algebra structure on the cohomology groups

$$\bigoplus_{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0}} \text{HH}^m(A, \Omega^p(A)).$$

We also recall that the bullet product  $\bullet$  has the following property (cf. [Wang2, Proposition 4.9]): for any  $f \in \text{HH}^m(A, \Omega^p(A))$  and  $g \in \text{HH}^n(A, \Omega^q(A))$ ,

$$f \cup g - (-1)^{(m-p)(n-q)} g \cup f = (-1)^{m-p} \delta(g \bullet f) = -(-1)^{(m-p-1)(n-q)} \delta(f \bullet g). \quad (8)$$

In the following, we will give a derived interpretation of Identity (8). Now let us fix two elements  $f \in \text{HH}^m(A, \Omega^p(A))$  and  $g \in \text{HH}^n(A, \Omega^q(A))$ , where  $m, n \in \mathbb{Z}_{>0}$  and  $p, q \in \mathbb{Z}_{\geq 0}$ . Let us consider the following chain complex (denoted by  $C_1(f, g)$ ) associated to  $f$  and  $g$ ,

$$\begin{array}{c} \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \oplus \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n-1} \oplus \text{Bar}_*(\Omega^{p+q}(A)) \otimes \epsilon_{m+n-2} \\ \begin{array}{c} \xrightarrow{\theta_1(f)} \\ \xrightarrow{\theta_2(g)} \\ \xrightarrow{\theta_1(\Omega^q(\theta_1(f)))} \\ \xrightarrow{\theta_2(\Omega^p(\theta_2(g)))} \end{array} \end{array} \quad (9)$$

it is straightforward to verify that this is a well-defined chain complex of  $A$ - $A$ -bimodules. We need to construct the following chain complex (denoted by  $C_2(f, g)$ ) of  $A$ - $A$ -bimodules associated to  $f$  and  $g$ .

$$\begin{array}{c}
 \xrightarrow{\theta_2(g)} \\
 \xrightarrow{\theta_1(f)} \\
 \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \oplus \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n-1} \oplus \Omega^{p+q}(A) \otimes \epsilon_{m+n-2}. \\
 \xrightarrow{\Omega^q(\theta_1(f))} \\
 \xrightarrow{\Omega^p(\theta_2(g))}
 \end{array} \tag{10}$$

We will also construct another chain complex (denoted by  $C_3(f, g)$ ) associated to  $f$  and  $g$ .

$$\begin{array}{c}
 \xrightarrow{\theta_1(g)} \\
 \xrightarrow{\theta_1(f)} \\
 \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \oplus \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n-1} \oplus \Omega^{p+q}(A) \otimes \epsilon_{m+n-2} \\
 \xrightarrow{\Omega^q(\theta_2(f))} \\
 \xrightarrow{\Omega^p(\theta_2(g))} \\
 \xrightarrow{f \bullet g}
 \end{array} \tag{11}$$

From Identity (8), it follows that the complex  $C_3(f, g)$  above is indeed a well-defined complex of  $(A^{\text{op}} \otimes A)$ -modules.

**Remark 3.4.** Each of the three complexes  $C_1(f, g)$ ,  $C_2(f, g)$  and  $C_3(f, g)$  can be considered as differential graded right  $(A^{\text{op}} \otimes A \otimes R)$ -modules, where

$$R := R_{m-p-1} \otimes R_{n-q-1}$$

is the tensor algebra of the differential graded algebras  $R_{m-p-1}$  and  $R_{n-q-1}$ . Moreover we have the following.

**Lemma 3.5.** *For any  $f \in \text{HH}^m(A, \Omega^p(A))$  and  $g \in \text{HH}^n(A, \Omega^q(A))$ , we have*

$$C_1(f, g) \cong C_2(f, g) \cong C_3(f, g)$$

*in the  $R$ -relative bounded derived category  $\mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$ , where*

$$R := R_{m-p-1} \otimes R_{n-q-1}.$$

*Proof.* Let us prove the first isomorphism. Note that we have the following morphism of dg  $(A^{\text{op}} \otimes A \otimes R)$ -modules,

$$\tilde{\sigma}_{p+q} := \begin{pmatrix} \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \\ 0 & 0 & \text{id} & 0 \\ 0 & 0 & 0 & \sigma_{p+q} \end{pmatrix}$$

$$C_1(f, g) \xrightarrow{\tilde{\sigma}_{p+q}} C_2(f, g).$$

where

$$\sigma_{p+q} : \text{Bar}_*(\Omega^{p+q}(A)) \otimes \epsilon_{m+n-2} \rightarrow \Omega^{p+q}(A) \otimes \epsilon_{m+n-2}$$

is the canonical surjection. Here we remark that  $\tilde{\sigma}_{p+q}$  is a quasi-isomorphism between chain complexes of right  $A^{\text{op}} \otimes A$ -modules since so is  $\sigma_{p+q}$ . Next we will show that  $\tilde{\sigma}_{p+q}$  is also an  $R$ -relative quasi-isomorphism. It is equivalent to show that the mapping cone  $\text{Cone}(\tilde{\sigma}_{p+q})$  is  $R$ -relatively acyclic. By the construction of the mapping cone, we have

$$\text{Cone}(\tilde{\sigma}_{p+q}) = C_1(f, g)[1] \overset{\tilde{\sigma}_{p+q}}{\oplus} C_2(f, g).$$

Hence by Remark 2.1, we can construct a homotopy of complexes of  $R$ -modules between  $\text{id}_{\text{C}(\tilde{\sigma}_{p+q})}$  and zero. Namely, we have a morphism of underlying graded  $R$ -modules

$$\tilde{s} : \text{Cone}(\tilde{\sigma}_{p+q}) \rightarrow \text{Cone}(\tilde{\sigma}_{p+q})[-1]$$

such that

$$\tilde{s} \circ d + d \circ \tilde{s} = \text{id},$$

where  $\tilde{s}$  can be constructed naturally from the homotopy  $s_p$  defined in Remark 2.1. Hence we have that

$$\text{Cone}(\tilde{\sigma}_{p+q}) = 0$$

in  $\mathcal{K}(R)$ . So  $\tilde{\sigma}_{p+q}$  is an isomorphism of  $\mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$ .

So it remains to verify the second isomorphism. Similarly, let us construct a morphism of dg  $(A^{\text{op}} \otimes A \otimes R)$ -modules

$$s(f, g) : C_3(f, g) \rightarrow C_2(f, g)$$

as follows,

$$s(f, g) := \begin{pmatrix} \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \\ s(g) & 0 & \text{id} & 0 \\ 0 & 0 & \tilde{s}(f) & \text{id} \end{pmatrix}$$

where  $\tilde{s}(f) : A^{\otimes q+m+1} \rightarrow \Omega^{p+q}(A)$  sends

$$a_{1,q+m+1} \in A^{\otimes q+m+1}$$

to

$$d(s^{q-1}(f)(a_{1,q+m+1})) \in \Omega^{p+q}(A)$$

and  $s(g)$  is defined in Section 2.2. From the definition of  $f \bullet g$  and Identity (4), it follows that  $s(f, g)$  is a well-defined morphism of dg  $(A^{\text{op}} \otimes A \otimes R)$ -modules. Similar to the proof of the first isomorphism, we can show that  $s(f, g)$  is an  $R$ -relative quasi-isomorphism via the homotopy  $s_p$  defined in Remark 2.1. More precisely, we can construct a homotopy

$$\hat{s} : \text{Cone}(s(f, g)) \rightarrow \text{Cone}(s(f, g))[-1]$$

between the identity morphism and the zero morphism of the mapping cone  $\text{Cone}(s(f, g))$  and as follows,

$$\hat{s} := \begin{pmatrix} s_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_p & 0 & 0 & (1 \otimes g) \\ 0 & 0 & 0 & 0 & s_q & 0 & (\sigma(f)) \\ 0 & 0 & 0 & 0 & 0 & s_q & (f \otimes 1) \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{id} \end{pmatrix}$$

where

$$\begin{aligned} (1 \otimes g)(a_{1,p+r}) &:= \begin{cases} d(a_{1,p+1} \otimes g(a_{p+2,n+p+1})) & \text{if } r = n+1, \\ 0 & \text{otherwise;} \end{cases} \\ \sigma(f)(a_{1,q+r}) &:= \begin{cases} d(f(a_{1,m}) \otimes a_{m+1,m+q} \otimes 1) & \text{if } r = m, \\ 0 & \text{otherwise;} \end{cases} \\ (f \otimes 1)(a_{1,q+r}) &:= \begin{cases} d(a_1 f(a_{2,m+1}) \otimes a_{m+2,m+q+1} \otimes 1) & \text{if } r = m+1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By direct calculation, we have that

$$\widehat{s} \circ d + d \circ \widehat{s} = \text{id}_{\text{Cone}(s(f,g))}.$$

Thus

$$\text{Cone}(s(f,g)) = 0$$

in the homotopy category  $\mathcal{K}(R)$ . Therefore,

$$C_1(f,g) \cong C_2(f,g) \cong C_3(f,g)$$

in the  $R$ -relative derived category  $\mathcal{D}_R^b(A \otimes_k A^{\text{op}} \otimes R)$ . ■

Let us consider the tensor product  $C(f) \otimes_A C(g)$  of the dg right  $A \otimes R_{m-p-1}$ -module  $C(f)$  and the dg left  $A \otimes R_{n-q-1}$ -module  $C(g)$  defined in Remark 3.2. It is clear that  $C(f) \otimes_A C(g)$  is endowed with a dg right  $A^{\text{op}} \otimes A \otimes R$ -module structure in a natural way, where  $R := R_{m-p-1} \otimes R_{n-q-1}$ . Moreover, we have the following.

**Lemma 3.6.** *For any  $f \in \text{HH}^m(A, \Omega^p(A))$  and  $g \in \text{HH}^n(A, \Omega^q(A))$ , we have the isomorphisms*

$$C(f) \otimes_A C(g) \cong C_2(f,g) \cong C_3(f,g)$$

in the  $R$ -relative bounded derived category  $\mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$ ,

*Proof.* Let us write down the complex  $C(f) \otimes_A C(g)$ . For simplicity, here we denote

$$\begin{aligned} B_*(A) &:= \text{Bar}_*(A), \\ \Omega^p(A)_{m-1} &:= \Omega^p(A) \otimes \epsilon_{m-1}, \\ \Omega^q(A)_{n-1} &:= \Omega^q(A) \otimes \epsilon_{n-1}. \end{aligned}$$

Then  $C(f) \otimes_A C(g)$  is illustrated by the following diagram.

$$\begin{array}{ccc} & \xrightarrow{\text{id} \otimes Ag} & \\ & \searrow f \otimes_A \text{id} & \\ B_*(A) \otimes_A B_*(A) \oplus \Omega^p(A)_{m-1} \otimes_A B_*(A) \oplus B_*(A) \otimes_A (\Omega^q(A)_{n-1} \oplus \Omega^p(A)_{m-1} \otimes_A \Omega^q(A)_{n-1}) & \xrightarrow{f \otimes_A \text{id}} & \end{array}$$

$\xrightarrow{\text{id} \otimes Ag}$

Let us construct a morphism of dg right  $(A^{\text{op}} \otimes A \otimes R)$ -modules

$$t(f,g) : C_2(f,g) \rightarrow C(f) \otimes_A C(g)$$

as follows,

$$t(f, g) := \begin{pmatrix} \Delta_{0,0} & 0 & 0 & 0 \\ 0 & \tilde{\Delta}_{p,0} & 0 & 0 \\ 0 & 0 & \tilde{\Delta}_{0,q} & 0 \\ 0 & 0 & 0 & \text{id} \end{pmatrix}$$

where  $\tilde{\Delta}_{p,0}$  is a lift of the identity  $\text{id} : \Omega^p(A) \rightarrow \Omega^p(A)$ , more precisely,

$$\tilde{\Delta}_{p,0} : \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \rightarrow \Omega^p(A)_{m-1} \otimes_A \text{Bar}_*(A)$$

sends

$$a_{1,p+r+2} \in \text{Bar}_r(\Omega^p(A))$$

to

$$(d(a_{1,p+1} \otimes 1)) \otimes_A (1 \otimes a_{p+2,p+r+2})$$

for any  $r \in \mathbb{Z}_{\geq 0}$ , and similarly  $\tilde{\Delta}_{0,q}$  is a lift of the identity  $\text{id} : \Omega^q(A) \rightarrow \Omega^q(A)$ , that is,

$$\tilde{\Delta}_{0,q} : \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n-1} \rightarrow \text{Bar}_*(A) \otimes_A \Omega^q(A)_{n-1}$$

sends

$$a_{1,q+r+2} \in \text{Bar}_r(\Omega^q(A))$$

to

$$(a_{1,r+1} \otimes 1) \otimes_A (d(1 \otimes a_{r+2,p+r+2}))$$

for any  $r \in \mathbb{Z}_{\geq 0}$ . Clearly,  $t(f, g)$  is indeed a morphism of dg right  $(A^{\text{op}} \otimes A \otimes R)$ -modules. Moreover, we can prove that the mapping cone  $\text{Cone}(t(f, g))$  is isomorphic to zero in the homotopy category  $\mathcal{K}(R)$  via the homotopy  $s_p$  defined in Remark 2.1. Hence  $t(f, g)$  induces an isomorphism

$$C_2(f, g) \cong C(f) \otimes_A C(g)$$

in  $\mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$ . ■

## 4 $R$ -relative derived tensor product

Let  $k$  be a field and  $R$  be a commutative differential graded  $k$ -algebra. Let  $A$  and  $B$  be two associative  $k$ -algebras and  $X$  be a differential graded right  $(A \otimes_k B^{\text{op}} \otimes_k R)$ -module, then we have the  $R$ -relative derived tensor product induced by  $X$ , in the sense of [Deli]:

$$- \otimes_{R \otimes_k B}^{\mathbb{L}, R} X : \mathcal{D}_R(B \otimes R) \rightarrow \mathcal{D}_R(A \otimes R).$$

**Remark 4.1.** From [Kel2, Section 7], it follows that

$$X \otimes_{B \otimes_k R}^{\mathbb{L}, R} - \cong \mathbf{p}_{\text{rel}} X \otimes_{B \otimes_k R} -,$$

where  $\mathbf{p}_{\text{rel}} X$  is  $R$ -relatively quasi-isomorphic to  $X$  and  $R$ -relatively closed, i.e.

$$\text{Hom}_{\mathcal{K}(B \otimes R)}(\mathbf{p}_{\text{rel}} X, M) \cong \text{Hom}_{\mathcal{D}_R(B \otimes R)}(\mathbf{p}_{\text{rel}} X, M)$$

for any differential graded  $(B \otimes R)$ -module  $M$ . Recall that in [Kel2, Section 7.4], we have the isomorphism

$$\text{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes R, M) \cong \text{Hom}_{\mathcal{D}_R(B \otimes R)}(B \otimes R, M)$$

for any differential graded  $(B \otimes R)$ -module  $M$ , hence

$$(B \otimes R) \otimes_{B \otimes_k R}^{\mathbb{L}, R} - \cong (B \otimes R) \otimes_{B \otimes_k R} -.$$

**Lemma 4.2.** *Let  $X$  be a differential graded  $R$ -module and  $P$  be a bounded complex of (ordinary) projective  $B$ -modules. Then we have*

$$\mathrm{Hom}_{\mathcal{K}(B \otimes R)}(P \otimes X, M) \cong \mathrm{Hom}_{\mathcal{D}_R(B \otimes R)}(P \otimes X, M) \quad (12)$$

for any differential graded  $(B \otimes R)$ -module  $M$ . As a consequence, we have

$$(P \otimes X) \otimes_{B \otimes_k R}^{\mathbb{L}, R} - \cong (P \otimes X) \otimes_{B \otimes_k R} -$$

*Proof.* First we observe that

$$\mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M) \cong \mathrm{Hom}_{\mathcal{K}(R)}(X, M)$$

for any dg  $(B \otimes R)$ -module  $M$ . Next we claim that

$$\mathrm{Hom}_{\mathcal{D}_R(B \otimes R)}(B \otimes X, M) \cong \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M). \quad (13)$$

Indeed, recall that in [Kel2, Section 7.4], we have

$$\mathrm{Hom}_{\mathcal{D}_R(B \otimes R)}(B \otimes X, M) \cong \varinjlim_{s: M \rightarrow M'} \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M') \quad (14)$$

where  $s : M \rightarrow M'$  is an  $R$ -relative quasi-isomorphism of dg  $(B \otimes R)$ -modules. For any  $R$ -relative quasi-isomorphism  $s : M \rightarrow M'$ , by definition, we have that the mapping  $\mathrm{Cone}(s)$  is  $R$ -relative acyclic, that is,  $\mathrm{Cone}(s) = 0$  in  $\mathcal{K}(R)$ . Hence  $s$  is an isomorphism in  $\mathcal{K}(R)$ , so we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}(R)}(X, M) & \xrightarrow[\cong]{\mathrm{Hom}_{\mathcal{K}(R)}(X, s)} & \mathrm{Hom}_{\mathcal{K}(R)}(X, M') \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M) & \xrightarrow{\mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, s)} & \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M') \end{array}$$

From this commutative diagram, it follows that the bottom morphism is an isomorphism, namely,

$$\mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, s) : \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M').$$

Hence from the isomorphism in (14), we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_R(B \otimes R)}(B \otimes X, M) &\cong \varinjlim_{s: M \rightarrow M'} \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M') \\ &\cong \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(B \otimes X, M) \end{aligned}$$

So we have proved the isomorphism in (13). For any bounded complex  $P$  of projective  $B$ -modules, we will prove the isomorphism in (12) by induction on the length  $l(P)$  of  $P$ . From the isomorphism in (13), it follows that

$$\mathrm{Hom}_{\mathcal{D}_R(B \otimes R)}(P \otimes X, M) \cong \mathrm{Hom}_{\mathcal{K}(B \otimes R)}(P \otimes X, M),$$

for  $l(P) = 1$  and any differential graded  $(B \otimes R)$ -module  $M$ . Suppose that the isomorphism in (12) holds for any complex  $P$  of projective  $B$ -modules such that  $l(P) = n - 1$ .

We need to prove that Isomorphism (12) holds for any complex  $P$  such that  $l(P) = n$ . Since  $P$  is bounded, there exists a distinguished triangle in  $\mathcal{K}^b(B\text{-proj})$

$$P_0 \rightarrow P_1 \rightarrow P \rightarrow P_0[1]$$

such that  $l(P_0) < n$  and  $l(P_1) < n$ . Hence we have a distinguished triangle in  $\mathcal{K}(B \otimes R)$ ,

$$P_0 \otimes B \rightarrow P_1 \otimes B \rightarrow P \otimes X \rightarrow P_0[1] \otimes X.$$

From a distinguished triangle in  $\mathcal{K}(B \otimes R)$ , one can induce a long exact sequence,

$$\begin{aligned} \cdots \rightarrow \text{Hom}_{\mathcal{K}(B \otimes R)}(P_0 \otimes X[n], M) &\rightarrow \text{Hom}_{\mathcal{K}(B \otimes R)}(P \otimes X[n-1], M) \rightarrow \\ \text{Hom}_{\mathcal{K}(B \otimes R)}(P_1 \otimes X[n-1], M) &\rightarrow \text{Hom}_{\mathcal{K}(B \otimes R)}(P_0 \otimes X[n-1], M) \rightarrow \cdots \end{aligned} \quad (15)$$

Note that we have a canonical triangle functor

$$\mathcal{K}(B \otimes R) \rightarrow \mathcal{D}_R(B \otimes R),$$

hence we also have a distinguished triangle in  $\mathcal{D}_R(B \otimes R)$

$$P_0 \rightarrow P_1 \rightarrow P \rightarrow P_0[1]$$

which induces the following long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P_0 \otimes X[n], M) &\rightarrow \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P \otimes X[n-1], M) \rightarrow \\ \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P_1 \otimes X[n-1], M) &\rightarrow \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P_0 \otimes X[n-1], M) \rightarrow \cdots \end{aligned} \quad (16)$$

By induction hypothesis, we have the following isomorphisms

$$\text{Hom}_{\mathcal{K}(B \otimes R)}(P_0 \otimes X[n], M) \cong \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P_0 \otimes X[n], M)$$

and

$$\text{Hom}_{\mathcal{K}(B \otimes R)}(P_1 \otimes X[n], M) \cong \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P_1 \otimes X[n], M)$$

for any  $n \in \mathbb{Z}$  and any dg  $(B \otimes R)$ -module  $M$  (since  $l(P_0) < n$  and  $l(P_1) < n$ ). From the two long exact sequences in (15) and (16) above, it follows that

$$\text{Hom}_{\mathcal{K}(B \otimes R)}(P \otimes X, M) \cong \text{Hom}_{\mathcal{D}_R(B \otimes R)}(P \otimes X, M).$$

Therefore, we have finished the proof. ■

**Proposition 4.3.** *Let  $m, p \in \mathbb{Z}_{\geq 0}$  and  $f, g \in \text{HH}^m(A, \Omega^p(A))$ , then we have the following isomorphisms in  $\mathcal{D}_{R_{m-p-1}}(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ ,*

$$C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}}^{\mathbb{L}, R_{m-p-1}} C(\theta_2(g)) \cong C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}} C(\theta_2(g)) \cong C_6(f, g),$$

where  $C_6(f, g)$  is the following dg  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module,

$$\text{Bar}_*(A) \oplus \Omega^p(A) \otimes_A \Omega^p(A) \otimes \epsilon_{m+p-1}.$$

$\xrightarrow{f \otimes_A d + d \otimes_A g}$

*Proof.* First, let us prove the first isomorphism. It is sufficient to show that  $C(\theta_1(f))$  is relatively closed in  $\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ , that is,

$$\mathrm{Hom}_{\mathcal{K}(A \otimes R_{m-p-1})}(C(\theta_1(f)), M) \cong \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})}(C(\theta_1(f)), M)$$

for any dg  $(A \otimes R_{m-p-1})$ -module  $M$ . Consider the following canonical morphism of differential graded  $(A \otimes R_{m-p-1})$ -modules

$$\pi : C(\theta_1(f)) \rightarrow \text{Bar}_*(A),$$

where

$$\mathrm{Bar}_*(A) \cong \mathrm{Bar}_*(A) \otimes k$$

is considered as a dg  $(A \otimes R_{m-p-1})$ -module and  $k$  has the canonical  $R_{m-p-1}$ -module structure. Denote by  $Cone(\pi)$  the mapping cone of  $\pi$  in  $\mathcal{K}(A \otimes R_{m-p-1})$ . By the construction of the mapping cone, we have

$$\begin{array}{ccc} & \text{id}_{\text{Bar}_*(A)} & \\ & \curvearrowright & \\ \text{Cone}(\pi) & = & C(\theta_1(f))[1] \oplus \text{Bar}_*(A). \end{array} \quad (17)$$

Hence we have a distinguished triangle in  $\mathcal{K}(A \otimes R_{m-p-1})$ ,

$$C(\theta_1(f)) \xrightarrow{\pi} \mathrm{Bar}_*(A) \longrightarrow \mathrm{Cone}(\pi) \longrightarrow C(\theta_1(f))[1] \quad (18)$$

which induces a long exact sequence (since  $\mathcal{K}(A \otimes R_{m-p-1})$  is a triangulated category),

$$\begin{aligned} \cdots \rightarrow \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M) &\rightarrow \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(C(\theta_1(f)[n], M) \rightarrow \\ \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(Cone(\pi)[n-1], M) &\rightarrow \cdots \end{aligned} \quad (19)$$

where for simplicity, we denote  $\mathcal{K}(A \otimes R_{m-p-1})$  by  $\mathcal{K}_{R_{m-p-1}}$ . Note that we have a natural morphism of triangulated categories,

$$\mathcal{K}_{R_{m-p-1}} \rightarrow \mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1}).$$

Hence we have also a long exact sequence in  $\mathcal{D}_{R_{m-p-1}} := \mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ ,

$$\begin{aligned} \cdots \rightarrow \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M) &\rightarrow \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(C(\theta_1(f)[n], M) \rightarrow \\ \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(Cone(\pi)[n-1], M) &\rightarrow \cdots \end{aligned} \quad (20)$$

**Claim 4.4.**

$$\mathrm{Hom}_{\mathcal{D}_{R_{m-n-1}}}(\mathrm{Bar}_*(A)[n], M) \cong \mathrm{Hom}_{\mathcal{K}_{R_{m-n-1}}}(\mathrm{Bar}_*(A)[n], M)$$

for any  $n \in \mathbb{Z}$  and  $M \in \mathcal{K}_{R_{m-p-1}}$ . That is,  $\text{Bar}_*(A)[n]$  is relatively closed in  $\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ .

*Proof of Claim 4.4.* Let us consider the canonical morphism

$$\sigma : \mathrm{Bar}_*(A) \rightarrow A.$$



Clearly it is a morphism of  $\mathrm{dg} (A \otimes R_{m-p-1})$ -modules. Then we have a distinguished triangle in  $\mathcal{K}(A \otimes R_{m-p-1})$ ,

$$\mathrm{Bar}_*(A) \xrightarrow{\epsilon} A \longrightarrow \mathrm{Cone}(\sigma) \longrightarrow \mathrm{Bar}_*(A)[1].$$

Note that  $Cone(\sigma)$  is  $(A \otimes R_{m-p-1})$ -relative acyclic since we have a homotopy between  $\text{id}_{Cone(\sigma)}$  and zero,

$$Cone(\sigma) \xrightarrow{s_0} Cone(\sigma)[-1]$$

where  $s_0$  is defined in Remark 2.1. Hence  $\sigma : \text{Bar}_*(A) \rightarrow A$  is an isomorphism in  $\mathcal{K}(A \otimes R_{m-p-1})$  and thus it is also an isomorphism in  $\mathcal{D}_{R_{m-p-1}}$ . So we have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M) \\ \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(\epsilon[n], M) \downarrow \cong & & \cong \downarrow \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(\epsilon[n], M) \\ \mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(A[n], M) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(A[n], M) \end{array}$$

From Lemma 4.2, it follows that the bottom morphism in the diagram above is an isomorphism, so is the top morphism. That is, we have

$$\mathrm{Hom}_{\mathcal{K}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}_{R_{m-p-1}}}(\mathrm{Bar}_*(A)[n], M).$$

Therefore, from the long exact sequences in (19) and (20) we obtain that it is sufficient to prove that  $Cone(\pi)$  is relatively closed, in order to prove that  $C(\theta_1(f))$  is relatively closed in  $\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ .

Let us consider the following morphism of  $\mathcal{K}_{R_{m-p-1}}$ ,

$$\iota : \mathrm{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1] \rightarrow \mathrm{Cone}(\pi),$$

which is defined as follows,

$$\begin{array}{ccc} \mathrm{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1] & \xrightarrow{\iota} & \mathrm{Cone}(\pi) \\ \downarrow = & & \downarrow = \\ \mathrm{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1] & \xrightarrow{(0, \mathrm{id}, 0)} & \mathrm{Cone}(\pi) \end{array}$$

where we write  $Cone(\pi)$  as follows,

$$\begin{array}{ccc}
& \text{id}_{\text{Bar}_*(A)} & \\
& \curvearrowright & \\
\text{Bar}_*(A)[1] & \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1] & \oplus \text{Bar}_*(A). \\
& \curvearrowleft \theta_1(f)[1] &
\end{array} \tag{21}$$

Clearly,  $\iota$  is a well-defined morphism of  $\mathrm{dg} (A \otimes R_{m-p-1})$ -modules. We claim that

$$Cone(\iota) = 0$$

in  $\mathcal{K}_{R_{m-p-1}}$ , thus it follows that  $\iota$  is an isomorphism in  $\mathcal{K}_{R_{m-p-1}}$ . Indeed, first we can write down  $Cone(\iota)$  as follows

$$\begin{array}{ccccccc}
& & & & \text{id}_{\text{Bar}_*(A)} & & \\
& & & \nearrow^{\theta_1(f)[1]} & & \searrow & \\
\text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[2] & \oplus & \text{Bar}_*(A)[1] & \oplus & \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1] & \oplus & \text{Bar}_*(A). \\
& \searrow_{\text{id}[1]} & & & & & 
\end{array}$$

Let us construct a homotopy between morphisms  $\text{id}_{Cone(\iota)}$  and zero,

$$s := \begin{pmatrix} s_p & 0 & d \otimes 1 & 0 \\ 0 & s_0 & s(f)[1] & 0 \\ 0 & 0 & s_p & 0 \\ 0 & 0 & 0 & s_0 \end{pmatrix} : Cone(\iota) \longrightarrow Cone(\iota)[1],$$

where  $s(f) : \text{Bar}_*(A) \rightarrow \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}$  is defined in (3) and

$$d \otimes 1 : \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[2] \rightarrow \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1}[1]$$

is defined as follows,

$$(d \otimes 1)(a_{1,p+2+r}) := \begin{cases} d(a_{1,p+2}) \otimes 1 & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

It is clear that  $s$  is a well-defined morphism of underlying graded  $(A \otimes R_{m-p-1})$ -modules such that

$$s \circ d + d \circ s = \text{id}.$$

Hence  $s$  is a homotopy between  $\text{id}_{Cone(\iota)}$  and zero, so

$$Cone(\iota) = 0$$

in  $\mathcal{K}_{R_{m-p-1}}$  and  $\iota$  is an isomorphism in  $\mathcal{K}_{R_{m-p-1}}$  (hence it is also an isomorphism in  $\mathcal{D}_{R_{m-p-1}}$ .) So  $\iota$  induces the following commutative diagram,

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{K}_{R_{m-p-1}}}(\Omega^p(A) \otimes \epsilon_{m-1}[1], M) & \xrightarrow[\cong]{\iota} & \text{Hom}_{\mathcal{K}_{R_{m-p-1}}}(Cone(\pi), M) \\
\downarrow \cong & & \downarrow \\
\text{Hom}_{\mathcal{D}_{R_{m-p-1}}}(\Omega^p(A) \otimes \epsilon_{m-1}[1], M) & \xrightarrow[\cong]{\iota} & \text{Hom}_{\mathcal{D}_{R_{m-p-1}}}(Cone(\pi), M)
\end{array} \tag{22}$$

for any dg  $(A \otimes R_{m-p-1})$ -module  $M$ . From Lemma 4.2, it follows that the left vertical morphism in Diagram (22) is an isomorphism, which implies that the right vertical morphism is also an isomorphism, namely,

$$\text{Hom}_{\mathcal{K}_{R_{m-p-1}}}(Cone(\pi), M) \cong \text{Hom}_{\mathcal{D}_{R_{m-p-1}}}(Cone(\pi), M).$$

Hence  $Cone(\pi)$  is relatively closed in  $\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ , so  $C(\theta_1(f))$  is also relatively closed in  $\mathcal{D}_{R_{m-p-1}}(A \otimes R_{m-p-1})$ . Therefore we have

$$C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}}^{\mathbb{L}, R_{m-p-1}} C(\theta_2(g)) \cong C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}} C(\theta_2(g)).$$

It remains to verify the second isomorphism of the statement of the lemma. Observe that we can write

$$C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}} C(\theta_2(g))$$

as the following form

$$\begin{array}{ccc} & \theta_1(f) & \\ \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} & \xrightarrow{\quad} & \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m-1} \\ & \theta_2(g) & \end{array} \otimes_{A \otimes R_{m-p-1}} \quad (23)$$

We claim that (23) is isomorphic to the following differential graded  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module (denoted by  $C_4(f, g)$ ),

$$\begin{array}{ccc} & \theta_1(f) \otimes \tau + \tau \otimes \theta_2(g) & \\ \text{Bar}_*(A) \otimes_A \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A))[p] \otimes_A \text{Bar}_*(\Omega^p(A))[p] \otimes R_{m-p-1} & \xrightarrow{\quad} & \end{array} \quad (24)$$

where  $\tau : \text{Bar}_*(A) \rightarrow \text{Bar}_*(\Omega^p(A))[p]$  is the canonical projection. Indeed, Recall the definition of  $C(\theta_1(f))$  in (5) from Section 3.1, as a graded  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module,

$$C(\theta_1(f)) := \bigoplus_{i=0}^{p-1} (A^{\otimes i+2} \otimes k[i]) \bigoplus \text{Bar}_*(\Omega^p(A)) \otimes R_{m-p-1}[p].$$

So (just) as graded  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -modules, we have

$$\begin{aligned} & C(\theta_1(f)) \otimes_{A \otimes R_{m-p-1}} C(\theta_1(g)) \\ & \cong \left( \bigoplus_{i=0}^{p-1} (A^{\otimes i+2} \otimes k[i]) \bigoplus \text{Bar}_*(\Omega^p(A)) \otimes R_{m-p-1}[p] \right) \otimes_{A \otimes R_{m-p-1}} \\ & \quad \left( \bigoplus_{i=0}^{p-1} (A^{\otimes i+2} \otimes k[i]) \bigoplus \text{Bar}_*(\Omega^p(A)) \otimes R_{m-p-1}[p] \right) \\ & \cong \left( \bigoplus_{i=0}^{p-1} (A^{\otimes i+2}[i]) \otimes_A \bigoplus_{i=0}^{p-1} (A^{\otimes i+2}[i]) \right) \bigoplus \left( \bigoplus_{i=0}^{p-1} (A^{\otimes i+2}[i]) \otimes_A \text{Bar}_*(\Omega^p(A))[p] \right) \bigoplus \\ & \quad \left( \text{Bar}_*(\Omega^p(A))[p] \otimes_A \bigoplus_{i=0}^{p-1} (A^{\otimes i+2}[i]) \right) \bigoplus (\text{Bar}_*(\Omega^p(A))[p] \otimes_A \text{Bar}_*(\Omega^p(A))[p] \otimes R_{m-p-1}) \\ & \cong \text{Bar}_*(A) \otimes_A \text{Bar}_*(A) \bigoplus \text{Bar}_*(\Omega^p(A))[p] \otimes_A \text{Bar}_*(\Omega^p(A))[p] \otimes \epsilon_{m-p-1}. \end{aligned}$$

Then by the construction of the tensor product of dg modules, we have that the differential is given as the one in  $C_4(f, g)$ . So the claim holds.

Recall that we have a quasi-isomorphism

$$\Delta_{0,0} : \text{Bar}_*(A) \rightarrow \text{Bar}_*(A) \otimes_A \text{Bar}_*(A).$$

Using this quasi-isomorphism, we can construct the following differential graded  $(A^{\text{op}} \otimes A \otimes R_{m-p-1})$ -module (denoted by  $C_5(f, g)$ ),

$$\begin{array}{ccc} & (\theta_1(f) \otimes \tau + \tau \otimes \theta_2(g)) \circ \Delta_{0,0} & \\ \text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^p(A))[p] \otimes_A \text{Bar}_*(\Omega^p(A))[p] \otimes \epsilon_{m-p-1} & \xrightarrow{\quad} & \end{array} \quad (25)$$

Since  $\Delta_{0,0}$  is a homotopy equivalence of complexes of  $A^{\text{op}} \otimes A$ -modules (cf. Lemma 2.2), we have that  $C_5(f, g)$  is  $R_{m-p-1}$ -relative quasi-isomorphic to  $C_4(f, g)$ . Since  $\text{Bar}_*(\Omega^p(A)) \otimes_A \text{Bar}_*(\Omega^p(A))$  is quasi-isomorphic to  $\Omega^p(A) \otimes_A \Omega^p(A)$ ,  $C_5(f, g)$  is  $R_{m-p-1}$ -relative quasi-isomorphic to  $C_6(f, g)$ , that is, the following dg module,

$$\text{Bar}_*(A) \oplus \Omega^p(A) \otimes_A \Omega^p(A) \otimes \epsilon_{m+p-1}.$$

$f \otimes_A d + d \otimes_A g$

■

## 5 Singular infinitesimal deformation theory

In this section, we follow [Kel1] to develop a theory of singular infinitesimal deformations of modules. Let  $k$  be a field and  $R$  be an augmented commutative differential graded  $k$ -algebra and denote by  $\mathfrak{n}$  the kernel of the augmentation  $R \rightarrow k$ . We always suppose that  $\dim_k \mathfrak{n} < \infty$  throughout this section.

Let  $A$  be an associative  $k$ -algebra, then  $A^{\text{op}} \otimes A \otimes R$  is a differential graded  $R$ -algebra. Let us denote by  $\mathcal{D}^{b, \text{perf}}(A^{\text{op}} \otimes A)$  the full subcategory of  $\mathcal{D}^b(A^{\text{op}} \otimes A)$  consisting of objects  $X \in \mathcal{D}^b(A^{\text{op}} \otimes A)$  such that  $X$  is a complex of projective left  $A$ -modules and is also a complex of projective right  $A$ -modules. For instance,  $A \in \mathcal{D}^{b, \text{perf}}(A^{\text{op}} \otimes A)$ . We also denote by  $\mathcal{D}_R^{b, \text{perf}}(A^{\text{op}} \otimes A \otimes R)$  the full subcategory of  $\mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$  consisting of objects  $X \in \mathcal{D}_R^b(A^{\text{op}} \otimes A \otimes R)$  such that  $X$  is a complex of projective left  $A$ -modules and is also a complex of projective right  $A$ -modules. For instance,  $A \otimes R \in \mathcal{D}_R^{b, \text{perf}}(A^{\text{op}} \otimes A \otimes R)$ . Consider the forgetful functor

$$F : \mathcal{D}_R^{b, \text{perf}}(A^{\text{op}} \otimes A \otimes R) \rightarrow \mathcal{D}^{b, \text{perf}}(A^{\text{op}} \otimes A)$$

and the induction functor

$$- \otimes_R k : \mathcal{D}_R^{b, \text{perf}}(A^{\text{op}} \otimes A \otimes R) \rightarrow \mathcal{D}^{b, \text{perf}}(A^{\text{op}} \otimes A).$$

**Remark 5.1.** We remark that the induction functor  $- \otimes_R k$  is well-defined. More generally, let  $f : R \rightarrow S$  is a morphism of commutative dg  $k$ -algebras. Suppose that  $A$  is a dg  $k$ -algebra. Then we have a well-defined functor

$$- \otimes_R S : \mathcal{D}_R(A \otimes R) \rightarrow \mathcal{D}_S(A \otimes S)$$

since  $- \otimes_R S$  sends  $R$ -relative quasi-isomorphisms to  $S$ -relative quasi-isomorphisms.

Let

$$\pi : \mathcal{D}^{b, \text{perf}}(A^{\text{op}} \otimes A) \rightarrow \mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$$

be the canonical projection. Denote by  $\text{Ker}(\pi \circ F)$  the full subcategory consisting of objects  $X \in \mathcal{D}_R^{b, \text{perf}}(A \otimes_k R)$  such that

$$\pi F(X) \cong 0$$

and by  $\text{Ker}(\pi \circ (- \otimes_R k))$  the full subcategory consisting of objects  $X \in \mathcal{D}_R^{b, \text{perf}}(A^{\text{op}} \otimes A \otimes R)$  such that

$$\pi(X \otimes_R k) \cong 0.$$

Then define the  $R$ -relative singular category  $\mathcal{D}_{\text{sg},R}(A^{\text{op}} \otimes A)$  of  $A^{\text{op}} \otimes A$  as the Verdier quotient

$$\frac{\mathcal{D}_R^{b,\text{perf}}(A^{\text{op}} \otimes A \otimes R)}{\text{Ker}(\pi \circ F) \cap \text{Ker}(\pi \circ (- \otimes_R k))}.$$

**Remark 5.2.** If  $R = k$ , then  $\mathcal{D}_{\text{sg},R}(A^{\text{op}} \otimes A) = \mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ .

## 5.1 Definition of singular infinitesimal deformation

Let  $k$  be a field and  $A$  be an associative  $k$ -algebra. Let  $R$  be an augmented commutative differential graded  $k$ -algebra. Let  $\mathbf{n}$  the kernel of the augmentation  $R \rightarrow k$ . In this subsection, we assume that  $\mathbf{n}^2 = 0$ . For example,  $R = k[\epsilon_m]/\epsilon_m^2$ . Define the singular infinitesimal deformation of  $A$  as the pair  $(L, u)$ , where  $L$  is a dg  $(A^{\text{op}} \otimes A \otimes R)$ -module such that the following canonical projection is an isomorphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ , namely,

$$L \otimes_R \mathbf{n} \xrightarrow{\cong} L\mathbf{n}$$

in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  and

$$u : L \otimes_R k \rightarrow A$$

is an isomorphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ . We also define  $\mathcal{F}$  as the category whose objects are the singular infinitesimal deformations  $(L, u)$  of  $A$  and morphisms from  $(L, u)$  to  $(L', u')$  are given by morphisms

$$v : L \rightarrow L'$$

of  $\mathcal{D}_{\text{sg},R}(A^{\text{op}} \otimes A)$  such that

$$u' \circ (v \otimes_R \text{id}_k) = u.$$

That is, the following diagram commutes in  $\mathcal{D}_{\text{sg}}(A \otimes_k A^{\text{op}})$ :

$$\begin{array}{ccc} L \otimes_R k & \xrightarrow{u} & A \\ v \otimes_R \text{id}_k \downarrow & \nearrow u' & \\ L' \otimes_R k & & \end{array}$$

We denote by

$$\text{sgDefo}(A, R \rightarrow k)$$

the set of isomorphism classes of  $\mathcal{F}$  and denote by

$$\text{sgDefo}'(A, R \rightarrow k)$$

the set of isomorphism classes of weak singular deformations of  $A$ , i.e. dg  $(A^{\text{op}} \otimes A \otimes R)$ -modules  $L$  such that

$$L \otimes_R \mathbf{n} \cong L\mathbf{n}$$

in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  and  $L \otimes_R k$  is isomorphic to  $A$  in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ . Note that the group  $\text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  of automorphisms of  $A$  in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  acts on  $\text{sgDefo}(A, R \rightarrow k)$  via

$$(L, u) \cdot f = (L, f^{-1} \circ u)$$

and the forgetful map induces a bijection

$$\text{sgDefo}(A, R \rightarrow k) / \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A) \cong \text{sgDefo}'(A, R \rightarrow k).$$

Let  $(L, u)$  be an object of  $\mathcal{F}$ . Since  $L$  is a dg  $(A^{\text{op}} \otimes A \otimes R)$ -module, we have the exact sequence of dg  $A^{\text{op}} \otimes A$ -modules which splits as a sequence of dg  $k$ -modules:

$$0 \rightarrow L\mathbf{n} \rightarrow L \rightarrow L \otimes_R k \rightarrow 0.$$

Thus it gives rises to a canonical triangle of  $\mathcal{D}^b(A^{\text{op}} \otimes A)$

$$L\mathbf{n} \rightarrow L \rightarrow L \otimes_R k \rightarrow L\mathbf{n}[1].$$

Since  $L \otimes_R \mathbf{n} \cong L\mathbf{n}$  in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ , we have the distinguished triangle of  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$

$$L \otimes_R \mathbf{n} \longrightarrow L \longrightarrow L \otimes_R k \xrightarrow{\epsilon'} L \otimes_R \mathbf{n}[1].$$

Since  $\mathbf{n}^2 = 0$ , we have a canonical isomorphism of dg modules

$$L \otimes_R \mathbf{n} \cong (L \otimes_R k) \otimes_k \mathbf{n}.$$

Therefore, we can define a canonical morphism  $\epsilon(L, u)$  of  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  by the following commutative square.

$$\begin{array}{ccc} L \otimes_R k & \xrightarrow{\epsilon'} & L \otimes_R \mathbf{n}[1] \xrightarrow{\cong} (L \otimes_R k) \otimes_k \mathbf{n}[1] \\ \downarrow u & & \downarrow u \otimes_k \text{id}_{\mathbf{n}[1]} \\ A & \xrightarrow{\epsilon(L, u)} & A \otimes_k \mathbf{n}[1] \end{array}$$

We claim that the morphism  $\epsilon(L, u)$  only depends on the isomorphism class of  $(L, u)$  in the category  $\mathcal{F}$ . Indeed, let  $(L', u') \in \mathcal{F}$  such that there exists an isomorphism

$$v : (L, u) \rightarrow (L', u')$$

in  $\mathcal{F}$ . To simplify the notational burden we denote by  $v \otimes k$  the morphism  $v \otimes \text{id}_k$ , etc. Then we have the following commutative diagram in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ ,

$$\begin{array}{ccccc} L \otimes_R k & \xrightarrow{\epsilon'} & L \otimes_R \mathbf{n}[1] & \xrightarrow{\cong} & (L \otimes_R k) \otimes_k \mathbf{n}[1] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \downarrow v \otimes k & & \downarrow v \otimes_R \mathbf{n}[1] & & \downarrow (v \otimes_R k) \otimes_k \mathbf{n}[1] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ L' \otimes_R k & \xrightarrow{\epsilon''} & L' \otimes_R \mathbf{n}[1] & \xrightarrow{\cong} & (L' \otimes_R k) \otimes_k \mathbf{n}[1] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \downarrow u' & & \downarrow u' \otimes_k \mathbf{n}[1] & & \downarrow \cong \\ A & \xrightarrow[\epsilon(L, u)]{\epsilon(L', u')} & A \otimes_k \mathbf{n}[1] \end{array}$$

(Curved arrows on the left and right indicate isomorphisms  $u \cong u' \otimes_k \mathbf{n}[1]$  and  $u \otimes_k \mathbf{n}[1] \cong u'$  respectively.)

where the morphism  $v \otimes_R \mathbf{n}[1] : L \otimes_R \mathbf{n}[1] \rightarrow L' \otimes_R \mathbf{n}[1]$  is an isomorphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  since  $v : L \rightarrow L'$  is an isomorphism in  $\mathcal{D}_{\text{sg}, R}(A^{\text{op}} \otimes A)$ . So we have  $\epsilon(L, u) = \epsilon(L', u')$  and thus we obtain a map

$$\Phi : \text{sgDefo}(A, R \rightarrow k) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \mathbf{n}[1])$$

which sends  $(L, u)$  to  $\epsilon(L, u)$ .

We will also construct a map

$$\Psi : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) \rightarrow \text{sgDefo}(A, R_m \rightarrow k)$$

as follows in the case of  $R = R_m$ . Let  $f : A \rightarrow A \otimes_k \epsilon_m[1]$  be a morphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ . Take a representative  $f' \in \text{Ext}^{p+m+1}(A, \Omega^p(A))$ , where we choose  $p \in \mathbb{Z}_{>0}$  such that  $p + m + 1 > 0$ . Recall that we have a dg right  $(A^{\text{op}} \otimes A \otimes R_m)$ -module  $C(\theta_1(f'))$  (cf. Section 3.1).

$$C(\theta_1(f')) := \text{Bar}_*(A) \quad \overset{\theta_1(f')}{\curvearrowright} \quad \oplus \quad \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{p+m}.$$

Clearly, the canonical morphism

$$C(\theta_1(f')) \otimes_{R_m} \epsilon_m \rightarrow C(\theta_1(f')) \epsilon_m$$

is an isomorphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  since we have the following commutative diagram in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ :

$$\begin{array}{ccc} C(\theta_1(f')) \otimes_{R_m} \epsilon_m & \longrightarrow & C(\theta_1(f')) \epsilon_m \\ \downarrow \cong & & \downarrow \cong \\ \text{Bar}_*(A) \otimes \epsilon_m & \xrightarrow{\cong} & \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{p+m} \end{array}$$

We have an obvious isomorphism

$$u' : C(\theta_1(f')) \otimes_{R_m} k \rightarrow A$$

in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ . Hence we have  $(C(\theta_1(f')), u') \in \mathcal{F}$ , let us define

$$\Psi(f) = (C(\theta_1(f')), u).$$

We need to show that  $\Psi(f)$  is independent of the choice of the representative of  $f$ . Take another representative  $f'' \in \text{Ext}^{q+m+1}(A, \Omega^q(A))$ , where  $q \in \mathbb{Z}_{>0}$  is chosen such that  $q + m + 1 > 0$ . Without loss of generality, let us assume  $q > p$ . Since both  $f'$  and  $f''$  represent the same element  $f$ , we have a natural projection of dg  $(A \otimes A^{\text{op}} \otimes R_m)$ -modules

$$\rho : C(\theta_1(f')) \rightarrow C(\theta_1(f'')).$$

Next let us prove that  $\rho$  is an isomorphism in  $\mathcal{D}_{\text{sg}, R}(A^{\text{op}} \otimes A)$ . Denote the mapping cone of  $\rho$  by  $\text{Cone}(\rho)$ , which is the following dg  $(A^{\text{op}} \otimes A \otimes R_m)$ -module

$$\text{Cone}(\rho) := C(\theta_1(f'))[1] \quad \overset{\rho}{\curvearrowright} \quad \oplus \quad C(\theta_1(f'')).$$

Then we have a distinguished triangle in  $\mathcal{D}_{R_m}(A^{\text{op}} \otimes A \otimes R_m)$

$$C(\theta_1(f')) \xrightarrow{\rho} C(\theta_1(f'')) \longrightarrow \text{Cone}(\rho) \longrightarrow C(\theta_1(f'))[1].$$

On the other hand, we have the embedding  $\iota : B_{p,q} \rightarrow \text{Cone}(\rho)$ , where  $B_{p,q}$  is the following complex

$$0 \longrightarrow A^{\otimes q+2} \xrightarrow{d_q} A^{\otimes q+1} \xrightarrow{d_{q-1}} \dots \xrightarrow{d_{p+1}} A^{\otimes p+2} \longrightarrow 0.$$

By using the homotopy defined in Remark 2.1, we have that  $\iota$  is an  $R_m$ -relative quasi-isomorphism, that is, it is an isomorphism in  $\mathcal{D}_{R_m}(A^{\text{op}} \otimes A \otimes R_m)$ . Note that we have

$$B_{p,q} \otimes_{R_m} k \cong B_{p,q}$$

and  $B_{p,q} \cong 0$  in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ . So it follows that the projection  $\rho : C(\theta_1(f')) \rightarrow C(\theta_1(f''))$  is an isomorphism in  $\mathcal{D}_{\text{sg},R}(A^{\text{op}} \otimes A)$ . Clearly, we have the following commutative diagram.

$$\begin{array}{ccc} C(\theta_1(f')) \otimes_{R_m} k & \xrightarrow{\rho \otimes \text{id}} & C(\theta_1(f'')) \otimes_{R_m} k \\ \downarrow u' & \swarrow u'' & \\ A & & \end{array}$$

Hence we obtain the following identity in  $\text{sgDefo}(A, R_m \rightarrow k)$

$$(C(\theta_1(f')), u') = (C(\theta_1(f'')), u'').$$

Therefore  $\Psi(f)$  is independent of the choice of the representative of  $f$  and we have a well-defined map

$$\Psi : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) \rightarrow \text{sgDefo}(A, R_m \rightarrow k).$$

Note that we have  $\Phi \circ \Psi = \text{id}$ . Therefore we have the following proposition.

**Proposition 5.3.** *Let  $m \in \mathbb{Z}$  and  $R_m$  be the commutative differential graded algebra  $k[\epsilon_m]/(\epsilon_m^2)$  with the degree  $-m$  of  $\epsilon_m$  and the differential  $d = 0$ . Then the map*

$$\Psi : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) \rightarrow \text{sgDefo}(A, R_m \rightarrow k)$$

*is injective.*

*Proof.* This is an immediate corollary of the identity  $\Phi \circ \Psi = \text{id}$ . ■

Note that the group  $\text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  of automorphisms of  $A$  in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  acts on  $\text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_{m-1}[1])$  via

$$s \cdot f := s^{-1} \circ f \circ s$$

for  $s \in \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  and  $f \in \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A[m])$ . Therefore, we have an injection

$$\Psi' : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) / \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A) \hookrightarrow \text{sgDefo}'(A, R_m \rightarrow k) \quad (26)$$

**Lemma 5.4.** *For any  $m \in \mathbb{Z}$ , the group  $\text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  acts trivially on  $\text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_{m-1}[1])$ . Namely, the following natural map is bijective.*

$$\text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_{m-1}[1]) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_{m-1}[1]) / \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A).$$



*Proof.* Let  $s \in \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  and  $f \in \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes \epsilon_{m-1}[1])$ , we need to show that

$$s^{-1} \circ f \circ s = f.$$

Note that

$$\text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A) \hookrightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A)$$

is an embedding. Since the composition of morphisms in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  corresponds to the cup product in  $\text{HH}_{\text{sg}}^*(A, A)$ , we have

$$\begin{aligned} s^{-1} \circ f \circ s &= s^{-1} \cup f \cup s \\ &= f \cup s^{-1} \cup s \\ &= f, \end{aligned}$$

where the second identity comes from the fact that the cup product is graded commutative. Therefore we have shown that the action of  $\text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A)$  is trivial and as a consequence, the canonical map is bijective

$$\text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A[m]) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A[m]) / \text{Aut}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A).$$

■

**Remark 5.5.** Combining Proposition 5.3, Lemma 5.4 and Formula (26), we obtain a natural embedding

$$\Psi' : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) \hookrightarrow \text{sgDefo}'(A, R_m \rightarrow k).$$

**Lemma 5.6.**  $\Psi'$  is a monoid morphism where the monoid structure on  $\text{sgDefo}(A, R \rightarrow k)$  is induced from the monoidal structure of  $\mathcal{D}_{\text{sg}, R}(A^{\text{op}} \otimes A)$ .

*Proof.* Take  $f, g \in \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1])$ , then they can be represented by two elements  $f', g' \in \text{HH}^{m+1+p}(A, \Omega^p(A))$ . From Proposition 4.3, it follows that

$$\begin{aligned} \Psi'(f) \otimes_{A \otimes R_m} \Psi'(g) &\cong C(\theta_1(f')) \otimes_{A \otimes R_m} C(\theta_2(g')) \\ &\cong C_6(f', g'), \end{aligned}$$

where we recall that  $C_6(f', g')$  is the following dg  $(A^{\text{op}} \otimes A \otimes R_m)$ -module,

$$\begin{array}{c} \xrightarrow{f' \otimes_A d + d \otimes_A g'} \\ \text{Bar}_*(A) \oplus \Omega^p(A) \otimes_A \Omega^p(A) \otimes \epsilon_{m+p-1}. \end{array}$$

Note that we have a natural morphism

$$\mu : \Omega^p(A) \otimes_A \Omega^p(A) \rightarrow \Omega^{2p}(A)$$

induced from the canonical isomorphism

$$A^{\otimes p+1} \otimes_A A^{\otimes p+1} \rightarrow A^{\otimes 2p+1}.$$

We have the following morphism of dg  $(A^{\text{op}} \otimes A \otimes R_m)$ -modules,

$$\begin{array}{ccc} \text{Bar}_*(A) \oplus \Omega^p(A) \otimes_A \Omega^p(A) \otimes \epsilon_{m+p-1} & \xrightarrow{\tilde{\mu}} & \text{Bar}_*(A) \oplus \Omega^{2p}(A) \otimes \epsilon_{m+p-1}. \\ \xrightarrow{f' \otimes_A d + d \otimes_A g'} & & \xrightarrow{\Omega^p(\theta_1(f' + g'))} \end{array} \quad (27)$$

Since  $\mu$  is an isomorphism in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ , it follows that  $\tilde{\mu}$  is an isomorphism in  $\mathcal{D}_{\text{sg}, R_m}(A^{\text{op}} \otimes A)$ . Note that the dg module in the right hand side of (27) is a representative of  $\Psi'(f + g)$ . Therefore we have shown that  $\Psi'$  is a monoid morphism. ■

## 5.2 Definition of the Lie algebra of an algebraic group

Let  $k$  be a field and  $A$  be an associative  $k$ -algebra. Denote by  $\mathbf{cdg}k$  the category of finite-dimensional augmented commutative dg  $k$ -algebras. We define a functor associated to  $A$ ,

$$\mathrm{sgDPic}_A : \mathbf{cdg}k \rightarrow \{\text{groups}\}$$

which sends  $R \in \mathbf{cdg}k$  to the  $R$ -relatively singular derived Picard group  $\mathrm{sgDPic}_A(R)$ .

$$\mathrm{sgDPic}_A(R) := \{L \in \mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A) \mid \text{there exist } L' \in \mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A) \text{ such that} \\ L \otimes_{A \otimes R}^{\mathbb{L},R} L' \cong L' \otimes_{A \otimes R}^{\mathbb{L},R} L \cong A \otimes R \text{ in } \mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A)\} / \sim.$$

where  $\sim$  means the isomorphisms in  $\mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A)$ . Clearly,  $\mathrm{sgDPic}_A(R)$  is a well-defined group and any morphism  $f : R \rightarrow S$  in  $\mathbf{cdg}k$  induces a group homomorphism

$$\mathrm{sgDPic}_A(f) : \mathrm{sgDPic}_A(R) \rightarrow \mathrm{sgDPic}_A(S)$$

which sends  $L \in \mathrm{sgDPic}_A(R)$  to  $L \otimes_R S \in \mathrm{sgDPic}_A(S)$ . Hence we have defined a generalized algebraic group. Let us consider the generalized Lie algebra

$$\mathrm{Lie} \, \mathrm{sgDPic}_A^m := \mathrm{Ker}(\mathrm{sgDPic}_A(R_m) \rightarrow \mathrm{sgDPic}_A(k)),$$

namely, we have

$$\mathrm{Lie} \, \mathrm{sgDPic}_A^m := \{L \in \mathrm{sgDPic}_A(R_m) \mid \text{such that } L \otimes_{R_m} k \cong A \text{ in } \mathcal{D}_{\mathrm{sg}}(A^{\mathrm{op}} \otimes A)\}.$$

Let  $\mathbf{n}$  be a dg  $k$ -module such that  $\dim_k \mathbf{n} < \infty$ . Let  $R = k \oplus \mathbf{n}$  denote the augmented commutative dg algebra with  $\mathbf{n}^2 = 0$ . Define

$$G(\mathbf{n}) := \mathrm{Im} \left( \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^{\mathrm{op}} \otimes A)}(A, A \otimes \mathbf{n}[1]) \hookrightarrow \mathrm{sgDefo}'(A, R \rightarrow k) \right).$$

**Remark 5.7.** Note that we have

$$G(\epsilon_m) \hookrightarrow \mathrm{Lie} \, \mathrm{sgDPic}_A^m.$$

From Lemma 4.3, it follows that  $\Psi'$  is a monoid morphism. Hence  $G(\mathbf{n})$  has a  $k$ -vector space structure inherited from that of  $\mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^{\mathrm{op}} \otimes A)}(A, A \otimes \mathbf{n}[1])$ . We will define a Lie bracket on the total space

$$\bigoplus_{m \in \mathbb{Z}} G(\epsilon_m)$$

as follows. Let  $(L_1, u_1)$  and  $(L_2, u_2)$  represent elements of  $G(\epsilon_m)$  and  $G(\epsilon_n)$ , respectively. Let  $U_i$  be the image of  $L_i$  in  $\mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A)$  where  $R = R_m \otimes R_n$ . Note that  $U_i$  are invertible objects of the monoidal category  $\mathcal{D}_{\mathrm{sg},R}(A^{\mathrm{op}} \otimes A)$  (cf. Lemma 5.8). Let  $V$  denote the commutator of  $U_1$  with  $U_2$ . Then the following Proposition 5.9 shows that  $V$  is an element of  $G(\epsilon_m \otimes \epsilon_n)$ . Let us define

$$[L_1, L_2] := V \in G(\epsilon_{m+n}).$$

**Lemma 5.8.** *Let  $m, n \in \mathbb{Z}$  and  $f \in \mathrm{HH}_{\mathrm{sg}}^{m+1}(A, A)$ . Set*

$$\widehat{\Psi(f)} := \Psi(f) \otimes_{R_m} (R_m \otimes R_n) \in \mathcal{D}_{\mathrm{sg}, R_m \otimes R_n}(A^{\mathrm{op}} \otimes A).$$

*Then we have the following isomorphism in the tensor category  $\mathcal{D}_{\mathrm{sg}, R_m \otimes R_n}(A^{\mathrm{op}} \otimes A)$ ,*

$$\widehat{\Psi(f)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(-f)} \cong A \otimes R_m \otimes R_n.$$

*Proof.* Since we have the following isomorphism in  $\mathcal{D}_{\text{sg}, R_m \otimes R_n}(A^{\text{op}} \otimes A)$ ,

$$\begin{aligned} \widehat{\Psi(f)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(-f)} &\cong (\Psi(f) \otimes_{R_m} (R_m \otimes R_n)) \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} (\Psi(-f) \otimes_{R_m} (R_m \otimes R_n)) \\ &\cong (\Psi(f) \otimes_{R_m} (R_m \otimes R_n)) \otimes_{A \otimes R_m \otimes R_n} (\Psi(-f) \otimes_{R_m} (R_m \otimes R_n)) \\ &\cong (\Psi(f) \otimes_{A \otimes R_m} \Psi(-f)) \otimes R_n \\ &\cong A \otimes R_m \otimes R_n \end{aligned}$$

where the second isomorphism is because of the fact that  $\widehat{\Psi(f)}$  is  $(R_m \otimes R_n)$ -relatively closed and the forth isomorphism comes from Proposition 4.3.  $\blacksquare$

**Proposition 5.9.** *Let  $f \in \text{HH}_{\text{sg}}^{m+1}(A, A)$  and  $g \in \text{HH}_{\text{sg}}^{n+1}(A, A)$ . Then we have that the commutator of  $\Psi(f)$  and  $\Psi(g)$ ,*

$$[\Psi(f), \Psi(g)] := \widehat{\Psi(f)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(g)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(-f)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(-g)}$$

*equals to  $\widehat{\Psi([f, g])}$  in  $G(\epsilon_m \otimes \epsilon_n)$ , where*

$$\widehat{\Psi(f)} := \Psi(f) \otimes_{R_m} (R_m \otimes R_n),$$

$$\widehat{\Psi(g)} := \Psi(g) \otimes_{R_n} (R_m \otimes R_n),$$

*$[f, g]$  is the Gerstenhaber bracket (cf. Definition 7) and*

$$\widehat{\Psi([f, g])} := \Psi([f, g]) \otimes_{R_{m+n}} (R_m \otimes R_n).$$

*Proof.* First, note that  $\Psi(f) = C(\theta_1(f))$ . Then from Lemma 5.8, it follows that the identity

$$[\Psi(f), \Psi(g)] = \widehat{\Psi([f, g])}$$

is equivalent to the following identity

$$\widehat{\Psi(f)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(g)} = \widehat{\Psi([f, g])} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(g)} \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} \widehat{\Psi(f)}. \quad (28)$$

Next let us verify Identity (28). From Lemma 3.6, it follows that the left hand side of (28) equals to the following,

$$\text{LHS} \cong C_3(f, g).$$

The right hand side is the following,

$$\begin{aligned} \text{RHS} &\cong (C(\theta_1([f, g])) \otimes_{R_{m+n}} (R_m \otimes R_n)) \otimes_{A \otimes R_m \otimes R_n}^{\mathbb{L}, R_m \otimes R_n} C_3(g, f) \\ &\cong (C(\theta_1([f, g])) \otimes_{R_{m+n}} (R_m \otimes R_n)) \otimes_{A \otimes R_m \otimes R_n} C_3(g, f) \\ &\cong C(\theta_1([f, g])) \otimes_{A \otimes R_{m+n}} C_3(g, f) \end{aligned}$$

where the first isomorphism comes from Lemma 3.6 and the second isomorphism is because of the fact that  $C(\theta_1([f, g])) \otimes_{R_{m+n}} (R_m \otimes R_n)$  is  $(R_m \otimes R_n)$ -relatively closed. Now

let us compute  $C(\theta_1([f, g])) \otimes_{A \otimes R_{m+n}} C_3(g, f)$ . Note that this is  $(R_m \otimes R_n)$ -relatively quasi-isomorphic to the following dg right  $(A^{\text{op}} \otimes A \otimes R_m \otimes R_n)$ -modules,

$$\begin{array}{c}
\begin{array}{c} \xrightarrow{[f, g]} \\ (\text{Bar}_*(A) \oplus \Omega^{p+q}(A) \otimes \epsilon_{m+n+p+q}) \otimes_{A \otimes R_{m+n}} \end{array} \\
\begin{array}{c} \xrightarrow{\theta_1(f)} \\ \xrightarrow{\theta_1(g)} \\ \xrightarrow{\Omega^p(\theta_2(f))} \\ \xrightarrow{\Omega^q(\theta_2(g))} \\ \xrightarrow{g \bullet f} \end{array} \\
(\text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n+q} \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m+p} \oplus \Omega^{p+q}(A) \otimes \epsilon_{m+n+p+q})
\end{array}$$

which is isomorphic to the following module,

$$\begin{array}{c}
\begin{array}{c} \xrightarrow{\theta_1(f)} \\ \xrightarrow{\theta_1(g)} \\ \xrightarrow{\Omega^p(\theta_2(f))} \\ \xrightarrow{\Omega^q(\theta_2(g))} \\ \xrightarrow{[f, g] + g \bullet f} \end{array} \\
\text{Bar}_*(A) \oplus \text{Bar}_*(\Omega^q(A)) \otimes \epsilon_{n+q} \oplus \text{Bar}_*(\Omega^p(A)) \otimes \epsilon_{m+p} \oplus \Omega^{p+q}(A) \otimes \epsilon_{m+n+p+q} \\
\cong C_3(f, g).
\end{array}$$

Hence we have LHS = RHS. Therefore, we have completed the proof.  $\blacksquare$

**Corollary 5.10.** *Let  $A$  be an associative algebra over a field  $k$ . Then the isomorphisms*

$$\Psi_m : \text{Hom}_{\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)}(A, A \otimes_k \epsilon_m[1]) \rightarrow G(\epsilon_m)$$

*induce an isomorphism of graded Lie algebras between the singular Hochschild cohomology  $\text{HH}_{\text{sg}}^*(A, A)$  and the total space*

$$\bigoplus_{m \in \mathbb{Z}} G(\epsilon_m).$$

*Proof.* This is a direct result of Proposition 5.9.  $\blacksquare$

**Remark 5.11.** We do not know whether the generalized Lie algebra  $\text{LiesgDPic}_A^*$  is a graded Lie algebra. But however, from Corollary 5.10 it follows that the graded subspace

$$\bigoplus_{m \in \mathbb{Z}} G(\epsilon_m) \subset \text{LiesgDPic}_A^*$$

is indeed a graded Lie algebra.

## 6 The invariance under singular equivalence of Morita type with level

Let  $A$  and  $B$  be two finite-dimensional  $k$ -algebras over a field  $k$ . Let  ${}_A M_B$  and  ${}_B N_A$  be a finite dimensional  $A$ - $B$ -bimodule and a finite dimensional  $B$ - $A$ -bimodule, respectively. Let us recall from [Wang1] that  $({}_A M_B, {}_B N_A)$  defines a singular equivalence of Morita type with level  $l \in \mathbb{Z}_{\geq 0}$  if the following conditions are satisfied,

1.  $M$  is projective as a left  $A$ -module and a right  $B$ -module, respectively,
2.  $N$  is projective as a left  $B$ -module and a right  $A$ -module, respectively,
3. we have the following isomorphisms

$$M \otimes_B N \cong \Omega^l(A)$$

in  $(A^{\text{op}} \otimes A)\text{-}\underline{\text{mod}}$ , and

$$N \otimes_A M \cong \Omega^l(B)$$

in  $(B \otimes B^{\text{op}})\text{-}\underline{\text{mod}}$ .

**Remark 6.1.** Note that the tensor product

$$M \otimes_B - : \mathcal{D}_{\text{sg}}(B) \rightarrow \mathcal{D}_{\text{sg}}(A)$$

is an equivalence of triangulated categories with the quasi-inverse

$$[l] \circ (N \otimes_A -) : \mathcal{D}_{\text{sg}}(A) \rightarrow \mathcal{D}_{\text{sg}}(B).$$

Similarly, we have the following equivalence of triangulated categories

$$M \otimes_B - \otimes_B N : \mathcal{D}_{\text{sg}}(B \otimes B^{\text{op}}) \rightarrow \mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A).$$

**Lemma 6.2.** *Let  $({}_A M_B, {}_B N_A)$  define a singular equivalence of Morita type with level  $l \in \mathbb{Z}_{\geq 0}$ . Then the functor  $M \otimes_B - \otimes_B N$  induces an isomorphism of graded commutative rings,*

$$M \otimes_B - \otimes_B N : \text{HH}_{\text{sg}}^*(B, B) \rightarrow \text{HH}_{\text{sg}}^*(A, A)$$

with the inverse

$$N \otimes_A - \otimes_A M : \text{HH}_{\text{sg}}^*(A, A) \rightarrow \text{HH}_{\text{sg}}^*(B, B).$$

*Proof.* It is a direct result of the facts that  $M \otimes_B - \otimes_B N$  induces an isomorphism between  $\mathcal{D}_{\text{sg}}(B^{\text{op}} \otimes B)$  and  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$  and that the cup product in  $\text{HH}_{\text{sg}}^*(A, A)$  corresponds to the composition of morphisms in  $\mathcal{D}_{\text{sg}}(A^{\text{op}} \otimes A)$ .  $\blacksquare$

Next we will show that singular equivalences of Morita type with level preserve Gerstenhaber algebra structures of singular Hochschild cohomology rings. Namely, we will show the following main theorem.

**Theorem 6.3.** *Let  $k$  be a field. Let  $A$  and  $B$  be two finite dimensional  $k$ -algebras. Suppose that  $({}_A M_B, {}_B N_A)$  defines a singular equivalence of Morita type with level  $l \in \mathbb{Z}_{\geq 0}$ . Then the functor  $[l] \circ (M \otimes_B - \otimes_B N)$  induces an isomorphism of Gerstenhaber algebras between singular Hochschild cohomology rings  $\text{HH}_{\text{sg}}^*(A, A)$  and  $\text{HH}_{\text{sg}}^*(B, B)$ .*

*Proof.* Note that the tensor functor  $M \otimes_B - \otimes_B N$  induces an isomorphism of generalized algebraic groups,

$$[l] \circ (M \otimes_B - \otimes_B N) : \text{sgDPic}_B \rightarrow \text{sgDPic}_A.$$

So it induces an isomorphism of generalized Lie algebras

$$[l] \circ (M \otimes_B - \otimes_B N) : \text{Lie sgDPic}_B \rightarrow \text{Lie sgDPic}_A,$$

in particular, it also induces an isomorphism of graded Lie algebras

$$[l] \circ (M \otimes_B - \otimes_B N) : G_B \rightarrow G_A,$$

where we denote

$$G_A := \bigoplus_{m \in \mathbb{Z}} G(\epsilon_m).$$

On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} G_B & \xrightarrow{\begin{smallmatrix} [l] \circ (M \otimes_B - \otimes_B N) \\ \cong \end{smallmatrix}} & G_A \\ \uparrow \cong & & \uparrow \cong \\ \text{HH}_{\text{sg}}^*(B, B) & \xrightarrow{\begin{smallmatrix} [l] \circ (M \otimes_B - \otimes_B N) \\ \cong \end{smallmatrix}} & \text{HH}_{\text{sg}}^*(A, A) \end{array}$$

Since from Corollary 5.10 we have that the vertical morphisms are isomorphisms of graded Lie algebras, the functor  $[l] \circ (M \otimes_B - \otimes_B N)$  induces an isomorphism of Gerstenhaber algebras between  $\text{HH}_{\text{sg}}^*(B, B)$  and  $\text{HH}_{\text{sg}}^*(A, A)$ . ■

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